

MAS212 Linear Algebra I Exercises 11 (with solutions)

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Do not submit your solutions to this final coursework, which is not assessed and will not be marked. Solutions are available from the course web site.

1. For each of the following matrices A , find an orthogonal matrix P and a diagonal matrix Λ such that $P A P^T = \Lambda$.

$$(a) \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, (b) \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}, (c) \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}.$$

Solution

(a)

eigenvalues := -1, 3

$$\text{eigenvectors} := \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Normalizing the eigenvectors gives $P^T = \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix}$ and hence $P =$

$$\begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix}.$$

Λ is the diagonal matrix of eigenvalues

$$\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

Check:

$$P P^T = \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P A P^T = \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\sqrt{2} & -\frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$$

(b)

$$\text{eigenvalues} := 1, 6$$

$$\text{eigenvectors} := \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Normalizing the eigenvectors gives $P^T = \begin{bmatrix} \frac{2}{5}\sqrt{5} & -\frac{1}{5}\sqrt{5} \\ \frac{1}{5}\sqrt{5} & \frac{2}{5}\sqrt{5} \end{bmatrix}$ and hence $P =$

$$\begin{bmatrix} \frac{2}{5}\sqrt{5} & \frac{1}{5}\sqrt{5} \\ -\frac{1}{5}\sqrt{5} & \frac{2}{5}\sqrt{5} \end{bmatrix}.$$

Λ is the diagonal matrix of eigenvalues

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

Check:

$$P P^T = \begin{bmatrix} \frac{2}{5}\sqrt{5} & \frac{1}{5}\sqrt{5} \\ -\frac{1}{5}\sqrt{5} & \frac{2}{5}\sqrt{5} \end{bmatrix} \begin{bmatrix} \frac{2}{5}\sqrt{5} & -\frac{1}{5}\sqrt{5} \\ \frac{1}{5}\sqrt{5} & \frac{2}{5}\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$P A P^T = \begin{bmatrix} \frac{2}{5}\sqrt{5} & \frac{1}{5}\sqrt{5} \\ -\frac{1}{5}\sqrt{5} & \frac{2}{5}\sqrt{5} \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} \frac{2}{5}\sqrt{5} & -\frac{1}{5}\sqrt{5} \\ \frac{1}{5}\sqrt{5} & \frac{2}{5}\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

(c)

$$\text{eigenvalues} := 4, -2, -2$$

$$\text{eigenvectors} := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Call the eigenvectors x_1, x_2, x_3 . The last two eigenvalues are identical and the corresponding eigenvectors x_2, x_3 are linearly independent but not orthogonal, so replace the last eigenvector x_3 by a linear combination of the last two eigenvectors x_2, x_3 that is orthogonal to x_2 using

$$x_3 := x_3 - \frac{x_3 \cdot x_2}{x_2 \cdot x_2} x_2 \text{ to give:}$$

$$\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Normalizing the eigenvectors gives $P^T = \begin{bmatrix} \frac{1}{3}\sqrt{3} & -\frac{1}{2}\sqrt{2} & -\frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{3} & \frac{1}{2}\sqrt{2} & -\frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{3} & 0 & \frac{1}{3}\sqrt{6} \end{bmatrix}$ and hence $P =$

$$\begin{bmatrix} \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ -\frac{1}{6}\sqrt{6} & -\frac{1}{6}\sqrt{6} & \frac{1}{3}\sqrt{6} \end{bmatrix}.$$

Λ is the diagonal matrix of eigenvalues

$$\Lambda = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Check:

$$P P^T = \begin{bmatrix} \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ -\frac{1}{6}\sqrt{6} & -\frac{1}{6}\sqrt{6} & \frac{1}{3}\sqrt{6} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{3}\sqrt{3} & -\frac{1}{2}\sqrt{2} & -\frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{3} & \frac{1}{2}\sqrt{2} & -\frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{3} & 0 & \frac{1}{3}\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P \Lambda P^T =$$

$$\begin{bmatrix} \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} & \frac{1}{3}\sqrt{3} \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ -\frac{1}{6}\sqrt{6} & -\frac{1}{6}\sqrt{6} & \frac{1}{3}\sqrt{6} \end{bmatrix} \cdot \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{3}\sqrt{3} & -\frac{1}{2}\sqrt{2} & -\frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{3} & \frac{1}{2}\sqrt{2} & -\frac{1}{6}\sqrt{6} \\ \frac{1}{3}\sqrt{3} & 0 & \frac{1}{3}\sqrt{6} \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

► **Maple Solution**

2. (a) Find an orthogonal matrix whose first *row* is the vector $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ and check it.
 (b) Find an orthogonal matrix whose first *column* is the vector $\left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$ and check it.

Solution

An orthogonal matrix has orthonormal rows and columns.

(a) We need a vector in the direction orthogonal to the direction of the given vector, namely $(1, 2)$. The orthogonal direction must be $(2, -1)$ or $(-2, 1)$. Normalizing this vector and making

it the second row of a matrix gives $P_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$ or $P_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

$$\text{Check: } P_1 P_1^T = P_1^2 = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, P_2 P_2^T = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(b) We need a vector in the direction orthogonal to the direction of the given vector, namely $(1, 3)$. The orthogonal direction must be $(3, -1)$ or $(-3, 1)$. Normalizing this vector and making

it the second column of a matrix gives $P_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}$ or $P_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$.

$$\text{Check: } P_1 P_1^T = P_1^2 = \frac{1}{10} \begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, P_2 P_2^T = \frac{1}{10} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Maple Solution

3. Let A and B be symmetric matrices. Give a necessary and sufficient condition for AB to be symmetric.

Solution

If AB is symmetric then $(AB)^T = AB$. But $(AB)^T = B^T A^T = BA$ since A and B are symmetric. Hence, a necessary and sufficient condition for AB to be symmetric is that $BA = AB$, i.e. that A and B are square matrices of the same size *that commute*.

4. Show that QQ^T and $Q^T Q$ are both symmetric matrices, where Q is any $m \times n$ matrix.

Solution

$$(QQ^T)^T = (Q^T)^T Q^T = QQ^T \text{ and } (Q^T Q)^T = Q^T (Q^T)^T = Q^T Q.$$

5. Prove that $(A_1 A_2 \cdots A_n)^T = A_n^T A_{n-1}^T \cdots A_1^T$, where $A_1, A_2, \dots, A_n$ are any matrices such that the product $A_1 A_2 \cdots A_n$ is defined. (State clearly any results you use.)

Solution

Suppose the result is true for n . Then $(A_1 A_2 \cdots A_n A_{n+1})^T = A_{n+1}^T (A_1 A_2 \cdots A_n)^T$ because $(AB)^T = B^T A^T$ as proved in [Chapter 11](#)
 $= A_{n+1}^T A_n^T A_{n-1}^T \cdots A_1^T$ by using the result for n .

Hence, the result is true for $n + 1$. It is clearly true for $n = 1$ so by induction it is true for all positive integer n .

6. (a) Let Q be any $m \times n$ real matrix and x any vector in $\mathbb{R}^n$. Prove that the scalar $x^T Q^T Q x \geq 0$.
- (b) Let Z be any $m \times n$ complex matrix and z be any vector in $\mathbb{C}^n$. Let z^+ denote the transposed complex conjugate of z , i.e. z^{*T} , and $\|z\| = \sqrt{z^+ z}$. Prove that the real scalar $z^+ Z^+ Z z \geq 0$ and that if Z is a square, non-singular $n \times n$ matrix and $z \neq 0$ then $z^+ Z^+ Z z > 0$.

Solution

(a) Let $y = Qx$, which is a vector in $\mathbb{R}^m$. Then $y^T y = \|y\|^2$ and $\|y\|^2 \geq 0$ by the norm axioms. Substituting the definition of y gives $0 \leq y^T y = (Qx)^T (Qx) = x^T Q^T Q x$.

(b) Let $y = Zz$, which is a vector in $\mathbb{C}^m$. Then $y^+ y = \|y\|^2$ and $\|y\|^2 \geq 0$ by the norm axioms. Substituting the definition of y gives $0 \leq y^+ y = (Zz)^+ (Zz) = z^+ Z^+ Z z$. If Z is a square, non-singular $n \times n$ matrix and $z \neq 0$ then $y \neq 0$, because if $y = 0$ then $z = Z^{-1} y \Rightarrow z = 0$, which is a contradiction. If $y \neq 0$ then $\|y\|^2 > 0$ by the norm axioms, hence $z^+ Z^+ Z z > 0$.